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# THE STABILITY OF CERTAIN DISCRETE-TIME VOLTERRA EQUATIONS†

V. B. KOLMANOVSKII

Moscow

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Lyapunov's second method is used to establish the conditions for the solutions of the discrete-time Volterra equations to be stable and bounded. The conditions are formulated directly in terms of the characteristics of the equations. © 1999 Elsevier Science Ltd. All rights reserved.

## 1. FORMULATION OF THE PROBLEM. DEFINITION OF STABILITY

Consider the scalar equation describing the change in the state $x_n$ of a system

$$x_{n+1} = -\sum_{i=0}^{n} a_{n,i}x_i, \quad n \geqslant 0 \tag{1.1}$$

where $n = 0, 1, \ldots$ is discrete time, $a_{n,i}$ is a given set of numbers defined for all $n \geqslant 0$, $0 \leqslant i \leqslant n$, and the initial state $x_0$ of the system is known.

Equations of type (1.1) are encountered in various applications, in numerical schemes for solving differential and integral equations and in convolution-type equations (i.e. when $a_{n,i} = a_{n-i}$)

$$x_{n+1} = -\sum_{i=0}^{n} a_{n-i}x_i = -\sum_{i=0}^{n} a_i x_{n-i} \tag{1.2}$$

which are widely used in renewal theory [2].

In all these cases it is important to consider the asymptotic properties of the solutions, and in particular, the problem of their stability with respect to perturbations of the initial state $x_0$.

*Definition.* System (1.1) is said to be stable if, for any $\varepsilon > 0$, $\delta(\varepsilon) > 0$ exists such that, if $|x_0| < \delta(\varepsilon)$, then $|x_n| < \varepsilon$ for all $n \geqslant 0$.

A stable system (1.1) is said to be asymptotically stable if $\lim_{n\to\infty} x_n = 0$ for any $x_0$ in the attraction domain of the trivial solution.

The method of Laplace transforms has been used [2] to obtain conditions for the asymptotic stability of Eqs (1.2) under the additional assumption that all the coefficients $a_i$ are of fixed sign and that the series formed by the $a_i$s is convergent. Modified theorems have been formulated for Lyapunov's second method in relation to the stability of solutions of Volterra equations, and they have been used to derive certain stability conditions for systems of type (1.1), on the assumption that the series of absolute values of the coefficients $a_{n,i}$ is convergent [3–6]. However, this assumption may sometimes turn out to be overly restrictive, since, for example, asymptotically stable systems of type (1.2) exist that do not satisfy it.

Below we will establish certain stability conditions that hold without the aforementioned assumption. They are obtained by constructing suitable Lyapunov functionals (i.e. positive definite functionals whose first difference along trajectories of the system is negative definite), and they are formulated in terms of sign-definite or monotone sequences of coefficients for scalar equations of convolution type (Section 2), for Eqs (1.1) (Section 3) and for the non-linear case (Sections 4 and 5). The functionals used in the paper are obtained by using a previously proposed procedure to construct them [6]. As an application, conditions are derived for the solutions of the perturbed equations to be bounded (Section 5).

We recall [7] that a sequence $a_i$ is said to be positive semi-definite if, for any $n$ and any finite sequence of numbers $x_0, \ldots, x_n$

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$$\sum_{i,j=0}^{n} a_{i+j}x_i x_j \geqslant 0$$

The definitions of positive and negative definite sequences are analogous.

Monotonicity of a sequence $a_i$ means that the differences between its terms have alternating signs. In particular, a sequence $a_i$ is said to be completely monotonic if, for all $i, j = 0, 1, \ldots$

$$(-1)^j \Delta^j a_i \geqslant 0, \quad \Delta a_i = a_{i+1} - a_i, \quad \Delta^0 a_i = a_i$$

An example of sequences of this type is the sequence of moments of a random variable [2].

## 2. SCALAR EQUATIONS OF CONVOLUTION TYPE

We will establish the conditions for Eqs (1.2) to be stable. Consider the following functional $V$, putting $a_{-1} = 2$

$$V = \sum_{i,j=0}^{n} a_{i+j-1}x_{n-i}x_{n-j} \tag{2.1}$$

Calculating the first difference $\Delta V = G - V$, we obtain

$$G = \sum_{i,j=0}^{n+1} a_{i+j-1}x_{n+1-i}x_{n+1-j} \tag{2.2}$$

Transform $G$ as follows:

$$G = \sum_{j=0}^{n+1} a_{j-1}x_{n+1}x_{n+1-j} + \sum_{i=1}^{n+1}\sum_{j=0}^{n+1} a_{i+j-1}x_{n+1-i}x_{n+1-j} = a_{-1}x_{n+1}^2 +$$

$$+2x_{n+1}\sum_{j=0}^{n} a_j x_{n-j} + \sum_{i,j=0}^{n} a_{i+j+1}x_{n-i}x_{n-j}$$

Note, moreover, that by Eqs (1.2)

$$x_{n+1}^2 = -x_{n+1}\sum_{i=0}^{n} a_i x_{n-i} \tag{2.3}$$

Equations (2.1)–(2.3) mean that

$$\Delta V = \sum_{i,j=0}^{n} (a_{i+j+1} - a_{i+j-1})x_{n-i}x_{n-j} \tag{2.4}$$

Now, depending on our assumptions concerning quadratic forms (2.1) and (2.4), we can formulate various stability criteria. For example, by previously known results [3], if the sequence $a_{i-1}$ is positive definite and $(a_{i+1}-a_{i-1})$ is negative definite, system (1.2) is asymptotically stable; however [3], this conclusion also holds under the following weaker assumptions concerning $a_i$.

*Theorem* 2.1. Suppose that for certain positive constants $c_1$, $c_2$ and $a_{-i} = 2$ the functionals (2.1) and (2.4) satisfy the estimates

$$V \geqslant c_1 x_n^2, \quad \Delta V \leqslant -c_2 x_n^2 \tag{2.5}$$

Then system (1.2) is asymptotically stable. But if $V \geqslant c_1 x_n^2$ and the sequence $(a_{i+1}-a_{-1})$ is negative definite, then system (1.2) is stable.

*Example* 2.1. Consider Eq. (1.2) with constant coefficients $a_i \equiv b$, $i \geqslant 0$

$$x_{n+1} = -b\sum_{i=0}^{n} x_{n-i}, \quad n \geqslant 0 \tag{2.6}$$

The series of the coefficients of Eq. (2.6) is divergent, that is, it is impossible to draw any conclusions as to stability on the basis of previous results that assume it to be convergent. At the same time, by Sylvester's criterion, the sequence $a_{-1} = 2$, $a_i \equiv b$, $i \geqslant 0$ is positive semi-definite if $0 \leqslant b < 2$. In addition, the form (2.1) in this case is $V = (2-b)x_n^2 + b(x_0 + \ldots + x_n)^2$, and by (2.6) we may conclude for the first difference that $\Delta V = -(2-b)x_n^2$. Thus, by Theorem 2.1, system (2.6) is asymptotically stable for $0 \leqslant b < 2$. This asymptotic stability condition is not only sufficient but also necessary. Indeed, if we introduce a new independent variable $y_n = b(x_0 + \ldots + x_{n-1})$, then Eq. (2.6) proves to be equivalent to a system of two equations $x_{n+1} = -bx_n - y_n$, $y_{n+1} = bx_n + y_n$. By the Schur–Cohn criterion, this system is asymptotically stable if and only if $0 \leqslant b < 2$.

Now let us put $X_{kn} = \Sigma_{j=k}^{n} x_j$ and turn to other stability conditions for system (1.2), obtained by using the functional

$$V = G_1 + G_2 + (2 - a_{-1})x_n^2; \quad G_1 = a_{n+1}X_{0n}^2, \quad G_2 = -\sum_{i=0}^{n}(a_{n+1-i} - a_{n-i})X_{in}^2 \tag{2.7}$$

$$\Delta V = \Delta G_1 + \Delta G_2 + (2 - a_{-1})\Delta x_n^2 \quad \Delta G_1 = a_{n+2}X_{0n+1}^2 - G_1,$$
$$\Delta G_2 = -\sum_{i=0}^{n+1}(a_{n+2-i} - a_{n+1-i})X_{in+1}^2 - G_2 \tag{2.8}$$

After transforming, we obtain

$$\Delta G_1 = (a_{n+2} - a_{n+1})X_{0n+1}^2 + a_{n+1}x_{n+1}(x_{n+1} + 2X_{0n})$$
$$\Delta G_2 = G_3 - \sum_{i=0}^{n+1}(a_{n+2-i} - 2a_{n+1-i} + a_{n-i})X_{in+1}^2$$
$$G_3 = \sum_{i=0}^{n}(a_{n+1-i} - a_{n-i})X_{in}^2 - \sum_{i=0}^{n+1}(a_{n+1-i} - a_{n-i})X_{in+1}^2 = G_4 - x_{n+1}^2(a_0 - a_{-1}) \tag{2.9}$$
$$G_4 = -x_{n+1}\sum_{i=0}^{n}(a_{n+1-i} - a_{n-i})(x_{n+1} + 2X_{in}) = -x_{n+1}^2(a_{n+1} - a_0) -$$
$$-2a_{n+1}x_{n+1}X_{0n} + 2x_{n+1}\sum_{i=0}^{n}a_{n-i}x_i$$

Relations (2.8), (2.9) and (2.4) mean that

$$\Delta V = (a_{n+2} - a_{n+1})X_{0n+1}^2 - \sum_{i=0}^{n+1}(a_{n+2-i} - 2a_{n+1-i} + a_{n-i})X_{in+1}^2 - (2 - a_{-1})x_n^2$$

We have thus proved the following theorem.

*Theorem* 2.2. Suppose that for some $\alpha_{-1} \in [0; 2)$ and all $i \geqslant 0$

$$a_i \geqslant 0, \quad a_{i+1} - a_i \leqslant 0, \quad a_{i+1} - 2a_i + a_{i-1} \geqslant 0; \quad i \geqslant 0 \tag{2.10}$$

Then system (1.2) is asymptotically stable.

Note that application of Theorem 2.2 to the test example 2.1 leads to the condition $0 \leqslant b < 2$ for the asymptotic stability of system (1.2), previously established with the help of Theorem 2.1.

## 3. EQUATIONS WITH VARIABLE COEFFICIENTS

We now present analogues of Theorems 2 .1 and 2.2 for equations of type (1.1). Set $a_{n,\,n+1} = 2$ for all $n \geqslant -1$ and continue $a_{n,j}$ into the domain $n \geqslant 0$, $i \leqslant -1$. Consider the functional

$$V = \sum_{i,j=0}^{n} a_{n-1,n-i-j}x_{n-i}x_{n-j} \tag{3.1}$$

After calculations analogous to those in Section 2, we obtain

$$\Delta V = \sum_{i,j=0}^{n} [a_{n,n-1-i-j} - a_{n-1,n-i-j}]x_{n-i}x_{n-j} \tag{3.2}$$

Thus, as in Theorem 2.1, we conclude that the following theorem holds.

*Theorem* 3.1. If forms (3.1) and (3.2) satisfy estimates (2.5), then system (1.1) is asymptotically stable.
Note that if $a_{n,i} = a_{n-i}$ forms (3.1) and (3.2) transform into (2.1) and (2.4), respectively.
We introduce certain sequences $a_{n-1,n}$ and $a_{n,-1}$, $n \geqslant 0$, assuming that

$$\sup_{n\geqslant 0} a_{n-1,n} < 2, \quad a_{n,-1} \geqslant 0, \quad a_{n+1,-1} - a_{n,-1} \leqslant 0 \tag{3.3}$$

Consider the functional

$$V = a_{n,-1}X_{0n}^2 - \sum_{i=0}^{n}(a_{n,i-1} - a_{n,i})X_{in}^2 + (2 - a_{n-1,n})x_n^2 \tag{3.4}$$

Calculations analogous to those in Section 2 give

$$\Delta V = -(2 - a_{n-1,n})x_n^2 + (a_{n+1,-1} - a_{n,-1})X_{0n+1}^2 - \sum_{i=0}^{n+1}(a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i})X_{in+1}^2$$

We have thus proved the following theorem.

*Theorem* 3.2. System (1.1) is asymptotically stable if inequalities (3.3) hold and in addition the following estimates hold for $0 \leqslant i \leqslant n+1$, $0 \leqslant j \leqslant n$, $n \geqslant 0$

$$a_{n,j-1} - a_{n,j} \leqslant 0, \quad a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i} \geqslant 0 \tag{3.5}$$

Note that if $a_{n,i} = a_{n-i}$ stability conditions (3.3) and (3.5) transform into conditions (2.10) of Theorem 2.2.

## 4. NON-LINEAR SYSTEMS

We will now establish certain stability conditions for the trivial solution of the scalar equation

$$x_{n+1} = -\sum_{i=0}^{n} a_{n-i}g(x_i), \quad n \geqslant 0 \tag{4.1}$$

where the function $g(x)$ is such that

$$g(0) = 0, \quad xg(x) > 0, \quad x \neq 0, \quad |g(x)| \leqslant |x| \tag{4.2}$$

Consider the functional

$$V = G_1 + G_2 \tag{4.3}$$
$$G_1 = 2x_n g(x_n) - a_{-1}g^2(x_n), \quad G_2 = \sum_{i,j=0}^{n} a_{i+j-1}g(x_{n-i})g(x_{n-j})$$

where the positive constant $a_{-1}$ is such that for all $x$

$$2 - a_{-1}\frac{g(x)}{x} \geqslant 0 \tag{4.4}$$

After calculations, we have

$$\Delta V = -g(x_n)(2x_n - a_{-1}g(x_n)) - \sum_{i,j=0}^{n}(a_{i+j-1} - a_{i+j+1})g(x_{n-i})g(x_{n-j}) \tag{4.5}$$

We have thus proved the following corollary.

*Corollary* 4.1. If conditions (4.2) are satisfied, a constant $a_{-1} > 0$ satisfying (4.4) exists, and inequalities (2.5) hold for the functionals $V$ and $\Delta V$ defined by (4.3) and (4.5), then the trivial solution of Eq. (4.1) is asymptotically stable.
We introduce the notation

$$\Gamma_{kn} = \sum_{j=k}^{n} g(x_j)$$

and consider the functional

$$V = G_1 + G_2 + G_3$$

$$G_1 = -\sum_{i=0}^{n}(a_{n+1-i} - a_{n-i})\Gamma_{in}^2, \quad G_2 = a_{n+1}\Gamma_{0n}^2, \quad G_3 = (2 - a_{-1})x_n g(x_n)$$

Through reductions similar to those performed in Section 2, we have

$$\Delta V = -(2 - a_{-1})x_n g(x_n) - a_{-1}g(x_{n+1})(x_{n+1} - g(x_{n+1})) -$$
$$-(a_{n+1} - a_{n+2})\Gamma_{0n+1}^2 - \sum_{i=0}^{n+1}(a_{n+2-i} - 2a_{n+1-i} + a_{n-i})\Gamma_{in+1}^2$$

We have thus proved the following corollary.

*Corollary* 4.2. Suppose estimates (4.2) and (2.10) hold and $a_{-1} \in [0; 2)$. Then the trivial solution of Eq. (4.1) is asymptotically stable.

*Example* 4.1. Suppose Eq. (4.1) has the form

$$x_{n+1} = -b(\sin x_1 + \ldots + \sin x_n)$$

By either of Corollaries (4.1) or (4.2), the trivial solution of this equation is asymptotically stable if $0 \leqslant b < 2$.

## 5. NON-LINEAR UNSTEADY SYSTEMS

Proceeding in the same way, we can derive the conditions for the stability of non-linear systems

$$x_{n+1} = -\sum_{i=0}^{n} a_{n,i} g(x_i) \tag{5.1}$$

We extend the definition of $a_{n,i}$ to $i = n + 1$ and $i \leqslant -1$, $n \geqslant 0$. Consider the functional

$$V = G_1 + G_2$$

$$G_1 = g(x_n)(2x_n - a_{n-1,n}g(x_n)), \quad G_2 = \sum_{i,j=0}^{n} a_{n-1,n-i-j}g(x_{n-i})g(x_{n-j})$$

After calculations, we obtain

$$\Delta V = \sum_{i,j=0}^{n}[a_{n,n-1-i-j} - a_{n-1,n-i-j}]g(x_{n-i})g(x_{n-j}) - G_1$$

*Corollary* 5.1. If relations (4.2) hold and the functionals $V$ and $\Delta V$ satisfy estimates (2.5), then the trivial solution of Eq. (5.1) is asymptotically stable.

Consider the functional

$$V = G_1 + G_2 + G_3$$

$$G_1 = a_{n,-1}\Gamma_{0n}^2, \quad G_2 = -\sum_{i=0}^{n}(a_{n,i-1} - a_{n,i})\Gamma_{in}^2, \quad G_3 = (2 - a_{n-1,n})x_n g(x_n)$$

Thus

$$\Delta V = (a_{n+1,-1} - a_{n,-1})G_{0n+1}^2 + G_4 - G_3 - a_{n,n+1}g(x_{n+1})(x_{n+1} - g(x_{n+1}))$$

$$G_4 = -\sum_{i=0}^{n+1}(a_{n+1,i-1} - a_{n+1,i} - a_{n,i-1} + a_{n,i})\Gamma_{in+1}^2$$

This proves the following corollary.

*Corollary* 5.2. Let the function $g(x)$ satisfy conditions (4.2) and suppose inequalities (3.3) and (3.5) are true. Then the trivial solution of Eq. (5.1) is asymptotically stable.

## 6. THE BOUNDEDNESS OF THE SOLUTIONS OF PERTURBED EQUATIONS

The stability conditions established above also enable us to establish certain conditions for the solutions of Volterra equations to be bounded, with various assumptions as to the nature of the perturbations.

We will first present some of these conditions for a system of type

$$x_{n+1} = -\sum_{i=n_0}^{n}(a_{n-i}x_i + b_{n,i}(x_i)); \quad n \geqslant n_0, \quad x_{n_0} = x_0 \tag{6.1}$$

where $a_i$ are given constants, $b_{n,i}$ are given functions and $n_0 \in N$, with $N$ a given index set.

We recall that system (6.1) is said to be: (1) bounded, if for any number $r > 0$ a number $\alpha(n_0, r)$ exists such that $|x_n| \leqslant \alpha(n_0, r)$ for all $n \geqslant n_0$ and $x_0$ such that $|x_0| \leqslant r$; (2) uniformly bounded relative to an initial value $n_0 \in N$, if $\alpha(n_0, r) \equiv \alpha(r)$.

*Theorem* 6.1. Suppose the assumptions of either of Theorems 2.1 or 2.2 are satisfied and the functions $b_{n,i}(x)$ satisfy inequalities $b_{n,i}(x)\ | \leqslant \gamma_{n,i}|x|$, where the constants $\gamma_{n,i} \geqslant 0$ are such that

$$\sum_{j=n_0}^{\infty}\sum_{r=j}^{\infty}\gamma_{r,j} < \infty \tag{6.2}$$

Then system (6.1) is bounded.

*Proof*. We express the solution of problem (6.1) in the form

$$x_n = R(n-n_0)x_0 + \sum_{i=n_0}^{n-1} R(n-i-1)\sum_{j=n_0}^{i} b_{i,j}(x_j); \quad n > n_0 \tag{6.3}$$

where $R(n)$ is a fundamental solution of Eq. (1.2), that is, a solution of Eq. (2.1) for $n > 0$ with initial condition $R(0) = 1$. By the assumptions of Theorem 6.1, the function $R(n)$ is bounded, that is, $|R(n)| \leqslant C$ for all $n \geqslant 0$ and some $C > 0$. Hence, and from (6.2) and (6.3), it follows that

$$|x_n| \leqslant C\left[|x_0| + \sum_{j=n_0}^{n-1}|x_j|\sum_{i=j}^{n-1}\gamma_{i,j}\right]$$

Applying the discrete version of the Gronwall–Bellman Lemma to the last inequality, we obtain

$$|x_n| \leqslant C|x_0|\exp\left[C\sum_{j=n_0}^{n-1}\sum_{i=j}^{n-1}\gamma_{i,j}\right] \tag{6.4}$$

In view of condition (6.2), it now follows from (6.4) that system (6.1) is bounded.

We will now find the condition for the solutions of the equations

$$x_{n+1} = -\sum_{i=n_0}^{n}a_{n-i}x_i + b_n; \quad n \geqslant n_0, \quad x_{n_0} = x_0 \tag{6.5}$$

to be bounded, where the perturbations $b_n$ are square summable.

*Theorem* 6.2. Suppose conditions (2.5) or the assumptions of Theorem 2.2 hold. Then system (6.5) is bounded if

$$\sum_{n=n_0}^{\infty} b_n^2 < \infty \tag{6.6}$$

*Proof*. Suppose conditions (2.5) are satisfied. Then, summing both sides of the second inequality of (2.5) over $n$ and using the fact that $V$ is non-negative, we conclude that any solution of Eq. (1.2) is square summable. In particular

$$\sum_{i=0}^{\infty} R^2(i) < \infty \tag{6.7}$$

It now follows from (6.6) and (6.7) and the representation

$$x_n = R(n-n_0)x_0 + \sum_{i=n_0}^{n-1} R(n-i-1)b_i$$

for the solutions of problem (6.5) that system (6.5) is bounded, since

$$|x_n| \leqslant C|x_0| + \left[\sum_{i=0}^{\infty} R^2(i) \sum_{j=n_0}^{\infty} b_j^2\right]^{1/2}$$

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